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Antinormal operators

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Introduction. Most results in operator theory in Hilbert space were, and still are, concerned with Hermitian/normal operators. Consequently, when dealing with more general operators, an interesting question is: how much does a given operator deviate from being normal? More accurately: what is, for a given operator T in a Hilbert space H, its distance to the set $\mathcal{N}$ of all normal operators? Such a question can be put for any non-empty set $\mathcal{M}\subset\mathcal{L}(H)$ and any $T\in\mathcal{L}(H)$: how can one calculate the distance from T to $\mathcal{M}$,

$$d(T,\mathcal{M}):=\inf\{\|T-A\|:\ A\in\mathcal{M}\},$$

in terms like $\|T\|$, $\sigma(T)$, $|T|$, $|T^*|$, ...? In the literature, the distances between an operator and the sets of Hermitian, positive, compact and unitary operators have been established in [4], [7], [9] and [3], respectively. The distance of an operator to the set of normal operators has been studied in [13], [5], [8], [10] and [11]. In these references the Hilbert space is mostly assumed to be separable. Because the null-operator is normal we always have $d(T,\mathcal{N})\leqq\|T\|$ for all $T\in\mathcal{L}(H)$. Operators of special interest are operators T with $d(T,\mathcal{N})$ as great as possible: $d(T,\mathcal{N})=\|T\|$. Such operators are called antinormal by R. B. HOLMES [8]. In this paper several equivalent descriptions of antinormality will be given without assuming separability of the Hilbert space.

Definitions and notations. Let H be an infinite dimensional Hilbert space. The dimension of a closed linear subspace D of H, dim D, is the cardinality of any orthonormal base of D. (See [14], page 45.) The sets of all normal, unitary, compact and invertible operators of $\mathcal{L}(H)$ are denoted by, resp., $\mathcal{N}$, $\mathcal{U}n$, $\mathcal{C}$ and $\mathcal{I}nv$. If $\mathcal{M}$ is a non-empty set of operators in $\mathcal{L}(H)$ and $T\in\mathcal{L}(H)$, then the distance between T and $\mathcal{M}$ is $d(T,\mathcal{M}):=\inf\{\|T-A\|:\ A\in\mathcal{M}\}$. An operator $T\in\mathcal{L}(H)$ is called

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antinormal ([8]) if $d(T, \mathscr{N})=\|T\|$ and *essentially antinormal* ([5] if $d(T, \mathscr{N}+\mathscr{C})=$ $=\|T\|$.

Denote the indicator function of a set A by 1_A. Let T be an operator. Denote its kernel by N_T, its range by R_T, its spectrum by $\sigma(T)$, its adjoint by T^* and its absolute value by $|T|=(T^*T)^{1/2}$. Define: $m(T):=\inf \sigma(|T|)$ and

$$n(T) := \inf\{x > 0\colon \dim 1_{[0,x]}(|T|)(H) \geqq \max(\aleph_0, \dim N_{T^*})\}.$$

Let ω be an infinite cardinal number. The sets $\mathscr{E}_\omega:=\{S\in\mathscr{L}(H)\colon \dim \bar{R}_S<\omega\}$ and $\mathscr{C}_\omega:=\operatorname{clo}\mathscr{E}_\omega$ are two-sided *-ideals of $\mathscr{L}(H)$. (See [12].) Let $\mathfrak{C}_\omega:=\mathscr{L}(H)/\mathscr{C}_\omega$, let $\pi_\omega\colon \mathscr{L}(H)\to\mathfrak{C}_\omega$ be the quotient map and $\|\cdot\|$ the quotient norm on $\mathfrak{C}_\omega$. For $A\in\mathscr{L}(H)$ let $\|A\|_\omega:=\|\pi_\omega(A)\|$. $\mathfrak{C}_\omega$ is a Banach algebra with identity. Let $\sigma_\omega(A):=$ $:=\sigma(\pi_\omega(A))$ $(A\in\mathscr{L}(H))$. Note that $\mathscr{C}_{\aleph_0}=\mathscr{C}$. An operator $T\in\mathscr{L}(H)$ is called *Fredholm* iff $\pi_{\aleph_0}(T)$ is invertible in $\mathfrak{C}_{\aleph_0}$, its *essential norm* is $\|T\|_e:=\|T\|_{\aleph_0}$ and its *essential spectrum* $\sigma_e(T):=\sigma_{\aleph_0}(T)$. Define $m_e(T):=\inf\sigma_e(|T|)$.

The index of an operator $T\in\mathscr{L}(H)$ is

$$\operatorname{ind} T := \begin{cases} 0 & \text{if } \dim N_T = \dim N_{T^*}, \\ \dim N_T - \dim N_{T^*} & \text{if } \dim N_T < \aleph_0 \text{ and } \dim N_{T^*} < \aleph_0, \\ \dim N_T & \text{if } \dim N_T \geqq \aleph_0 \text{ and } \dim N_{T^*} < \dim N_T, \\ -\dim N_T & \text{if } \dim N_{T^*} \geqq \aleph_0 \text{ and } \dim N_T < \dim N_{T^*}. \end{cases}$$

Observe that $\operatorname{ind} T = -\operatorname{ind} T^*$ for all T.

If $(X, \mathscr{B}, m)$ is a measure space, then M_h, $h\in L^\infty(m)$, denotes the multiplication operator $f\to fh$ in $\mathscr{L}(L^2(m))$.

First we prove an inequality on $d(T, \mathscr{N})$. For that purpose Theorem 6 of [3] is needed:

Theorem 1. *Let* $T\in\mathscr{L}(H)$. *Then*

$$d(T, \mathscr{U}_n) = \max(1-m(T), \|T\|-1) \quad \textit{if} \quad \operatorname{ind} T = 0,$$
$$d(T, \mathscr{U}_n) = \max(1+n(T), \|T\|-1) \quad \textit{if} \quad \operatorname{ind} T < 0.$$

The next theorem extends Theorem 1 of [10] to a non-separable Hilbert space. Its proof is similar to Izumino's with $m_e(T)$ replaced by $n(T)$ (which is the same in separable Hilbert spaces).

Theorem 2. *Let* $T\in\mathscr{L}(H)$. *Then*

$$d(T, \mathscr{N}) \leqq (\|T\|-m(T))/2 \quad \textit{if} \quad \operatorname{ind} T = 0,$$
$$n(T) \leqq d(T, \mathscr{N}) \leqq (\|T\|+n(T))/2 \quad \textit{if} \quad \operatorname{ind} T < 0.$$

From the first inequality it follows that an antinormal operator with index zero is trivial, it is the zero operator. Therefore in this paper we mostly consider operators with negative index.

Corollary 3. *Let* $T\in\mathcal{L}(H)$, $\operatorname{ind} T<0$. *The following conditions are equivalent.*

(i) *T is antinormal,*

(ii) $\|T\| = n(T)$,

(iii) $d(T, \mathcal{U}n) = 1+\|T\|$.

Proof. Combine Theorems 1 and 2.

Corollary 4. *Let* $T\in\mathcal{L}(H)$. *Then T is antinormal iff* $d(T, \mathcal{U}n)=1+\|T\|$.

Proof. Suppose $\operatorname{ind} T=0$ and $d(T, \mathcal{U}n)=1+\|T\|$. Then $T=0$ by Theorem 1 and T is antinormal. The remaining part follows from the remark after Theorem 2, Corollary 3 and the simple equalities $d(S, \mathcal{U}n)=d(S^*, \mathcal{U}n)$ $d(S, \mathcal{N})=d(S^*, \mathcal{N})$ and $\|S\|=\|S^*\|$ for all $S\in\mathcal{L}(H)$.

For separable Hilbert spaces the next theorem can be proved with the aid of Fredholm operators. (See [10], Theorem 2). For the general case we need two lemmas.

Lemma 5. *Let* $P\in\mathcal{L}(H)$ *be positive Hermitian and let* ω *be an infinite cardinal number. Then*

$$\|P\|_\omega = \inf\{\varepsilon > 0\colon \dim 1_{[\varepsilon, \infty)}(P)(H) < \omega\}.$$

Corollary. $\|P^2\|_\omega=\|P\|_\omega^2$.

Proof. Because of the Spectral Theorem we may assume, without loss of generality, that there exist a measure space $(X, \mathcal{B}, m)$ and a real function h in $L^\infty(m)$ such that $H=L^2(m)$ and $P=M_h$. Let $\varepsilon>0$ such that $1_{[\varepsilon, \infty]}(P)\in\mathcal{E}_\omega$. Let $k:=h\cdot 1_{\{x\in X: h(x)\geqq\varepsilon\}}\in L^\infty(m)$. Then $M_k\in\mathcal{C}_\omega$, $\|M_h-M_k\| \leqq \varepsilon$, so $\|P\|_\omega \leqq \inf\{\varepsilon > 0\colon \dim 1_{[\varepsilon, \infty)}(P)(H) < \omega\}$.

For the reverse inequality, let $\varepsilon>0$ such that $\dim 1_{[\varepsilon, \infty)}(P)(H)\geqq\omega$. Let $D:=1_{[\varepsilon, \infty)}(M_h)(H)$. Then D is a closed linear subspace of H and $\dim D\geqq\omega$. Define $j\colon X\to\mathbf{R}$ by

$$j(x) = h(x)^{-1} \quad\text{if}\quad h(x)\geqq\varepsilon;\quad j(x)=0 \quad\text{otherwise}\quad (x\in X).$$

Then $1_{[\varepsilon, \infty)}(M_h)=M_{jh}$ and $\|M_j\|\leqq\varepsilon^{-1}$. Let Q be the projection onto D. Then $PM_jQ=Q$.

Suppose $\|P\|_\omega<\varepsilon$. Then there exists an A in $\mathcal{E}_\omega$ such that $\|P-A\|<\varepsilon$. In the Hilbert space D the following inequalities hold: $\|QAM_jQ|_D-I_D\|_D=\|QAM_jQ|_D-Q|_D\|_D\leqq\|QAM_jQ-Q\|\leqq\|Q\|\cdot\|A-P\|\cdot\|M_j\|\cdot\|Q\|<1$. Thus, $QAM_jQ|_D\colon D\to D$ is invertible and $D\subset R_{QAM_jQ}\subset Q(R_A)$. Then $\omega\leqq\dim D\leqq\dim Q(R_A)\leqq\dim R_A<\omega$. Contradiction. Hence, $\|P\|_\omega\geqq\varepsilon$.

Lemma 6. *Let* $T\in\mathscr{L}(H)$, $\operatorname{ind} T<0$ *and* $\omega:=\max(\aleph_0, \dim N_{T^*})$. *Then* $n(T)\leqq\min\sigma_\omega(|T|)$.

Proof. Without loss of generality we may assume again that there exist a measure space $(X, \mathscr{B}, m)$ and a real function $h\in L^\infty(m)$ such that $H=L^2(m)$ and $|T|=M_h$. Let $0\leqq a<n(T)$, $a<b<n(T)$. From the definition of $n(T)$ it follows that $1_{[0,b)}(|T|)\in\mathscr{E}_\omega\subset\mathscr{C}_\omega$. Let

$$k:=(h-a)\cdot 1_{\{x\in X:\, h(x)>b\}}+1_{\{x\in X:\, h(x)\leqq b\}}\in L^\infty(m).$$

Then $k\geqq\min(b-a, 1)>0$, so M_k and $\pi_\omega(M_k)$ are invertible. Also, $M_k-M_{h-a}=$ $=M_{1-(h-a)}1_{[0,b]}(|T|)\in\mathscr{C}_\omega$. Thus, $\pi_\omega(M_{h-a})=\pi_\omega(M_k)$ is invertible and $a\notin\sigma_\omega(|T|)$. Because $\sigma_\omega(|T|)$ is compact, we can infer that $\min\sigma_\omega(|T|)\geqq n(T)$.

Theorem 7. *Let* $T\in\mathscr{L}(H)$ *be antinormal and* $\operatorname{ind} T<0$. *Then there exist* $\alpha>0$, *a non-surjective isometry* W *with* $\dim N_{W^*}\leqq\dim N_{T^*}$ *and a positive Hermitian contraction* $K\in\mathscr{L}(H)$ *such that* $T=\alpha W(I-K)$ *and* $K\in\mathscr{C}_\omega$ *with* $\omega:=$ $:=\max(\aleph_0, \dim N_{T^*})$. *Furthermore,* W *can be chosen such that* $\dim N_{W^*}=$ $=\dim N_{T^*}$ *if* $\dim N_{T^*}$ *is infinite.*

Proof. Let $\alpha:=\|T\|>0$. For the remaining part of the proof suppose $\|T\|=1$. Let $K:=I-|T|$. Then K is a positive Hermitian contraction. With the aid of polar decomposition we find a partial isometry V from $N_T^\perp$ to $N_{T^*}^\perp$ such that $T=V|T|$. Because $\dim N_T<\dim N_{T^*}$, V can be extended to a non-surjective isometry W with $T=W|T|$, $\dim N_{W^*}\leqq\dim N_{T^*}$ and in the case $\dim N_{T^*}\geqq\aleph_0$ even $\dim N_{W^*}=$ $=\dim N_{T^*}$. Then $T=\alpha W(I-K)$.

As T is antinormal and $\operatorname{ind} T<0$ we have (using Lemma 6): $1=\|T\|=n(T)\leqq$ $\leqq\min\sigma_\omega(|T|)\leqq\max\sigma_\omega(|T|)\leqq\max\sigma(|T|)\leqq\||T|\|=1$. Therefore, $\sigma_\omega(|T|)=\{1\}$ and $\sigma_\omega(K)=\{0\}$. Lemma 5 yields, by induction, that for all $n\in\mathbf{N}$, $\|K^{2^n}\|_\omega=\|K\|_\omega^{2^n}$. Let r be the spectral radius of $\pi_\omega(K)$ in $\mathfrak{C}_\omega$. Then $0=r=\lim_{n\to\infty}\|(\pi_\omega(K))^n\|^{1/n}=$ $=\lim_{n\to\infty}(\|K^n\|_\omega)^{1/n}=\lim_{n\to\infty}(\|K^{2^n}\|_\omega)^{2^{-n}}=\|K\|_\omega$. Thus, $K\in\mathscr{C}_\omega$.

Theorem 8. *Let* $\alpha\in\mathbf{R}$, $\alpha>0$ *and let* $W\in\mathscr{L}(H)$ *be a non-surjective isometry. Let* $\omega:=\max(\aleph_0, \dim N_{W^*})$ *and let* $K\in\mathscr{C}_\omega$ *be a positive Hermitian contraction. Let* $T:=\alpha W(I-K)$. *Then* $\alpha=\|T\|$ *and for all unitary* $U\in\mathscr{L}(H)$ *we have* $\sigma(UT)=\{z\in\mathbf{C}: |z|\leqq\|T\|\}$.

Proof. Without loss of generality we can take $\alpha=1$ and $U=I$. We have one simple inequality: $\|T\|\leqq\|W\|\cdot\|I-K\|\leqq 1$. For the remaining part of the proof we distinguish two cases.

Case 1. Suppose $\dim N_{W^*}\leqq\aleph_0$. This case is similar to Theorem 5 of [5]. Let $z\in\mathbf{C}$, $|z|<1$. Then $I-zW^*$ is invertible and therefore Fredholm with index 0.

We have $W^*(T-z)=I-K-zW^*$, so $W^*(T-z)$ is Fredholm and ind $W^*(T-z)=0$ by Theorem 5.20 of [2]. Similarly to the proof of Theorem 5.17 in [2] it follows that $\dim N_{T-z}<\aleph_0$ and R_{T-z} is closed.

— Suppose $\dim N_{(T-z)^*}<\aleph_0$. Then $T-z$ is Fredholm by Theorem 5.17 in [2], so W^* is too. By the additivity of the index for Fredholm operators (see [1], Theorem 2.1) we have $\operatorname{ind}(T-z)=\operatorname{ind} W^*(T-z)-\operatorname{ind} W^*=\operatorname{ind} W<0$.

— Suppose $\dim N_{(T-z)^*}\geqq\aleph_0$. Then $\operatorname{ind}(T-z)=-\dim N_{(T-z)^*}<0$.

Thus, ind $(T-z)<0$ and $\dim N_{(T-z)^*}>0$. Hence, $\bar{z}$ is an eigenvalue of T^* and $z\in\sigma(T)$. Then $\{z\in\mathbf{C}: |z|<1\}\subset\sigma(T)$ and $\|T\|\geqq 1$. We see that $\|T\|=1$ and $\sigma(T)=\{z\in\mathbf{C}: |z|\leqq 1\}$.

Case 2. Suppose $\dim N_{W^*}>\aleph_0$. Let $z\in\mathbf{C}$, $|z|<1$ and assume that $T-z$ has an inverse, A, in $\mathscr{L}(H)$. As $K\in\operatorname{clo}\mathscr{E}_\omega$ and $\|A\|>0$, there exists an $S\in\mathscr{E}_\omega$ with $\|K-S\|<\|A\|^{-1}$. Then $\|[W(I-S)-z]-[T-z]\|=\|W(K-S)\|\leqq\|W\|\cdot\|K-S\|<$ $<\|A\|^{-1}$, so $W(I-S)-z$ is invertible. Denote its inverse by D.

Let H_1 be the smallest closed linear subspace of H that contains R_{S^*} and is invariant under S, S^*, W, W^*, D and D^*. Then $\dim H_1\leqq\max(\aleph_0, \dim R_{S^*})<$ $<\omega=\dim N_{W^*}$, so $N_{W^*}\not\subset H_1$. Because $N_{W^*}=(N_{W^*}\cap H_1)\oplus(N_{W^*}\cap H_1^\perp)$ we have $N_{W^*}\cap H_1^\perp\neq\{0\}$. Choose $a\in N_{W^*}$, $a\perp H_1$, $a\neq 0$.

Let H_2 be the smallest closed linear subspace of $H_1^\perp$ that contains a and is invariant under S, S^*, W, W^*, D and D^*. This H_2 is separable. Momentarily, by a subscript "2" we indicate the restriction of an operator to H_2, viewed as an element of $\mathscr{L}(H_2)$. Then W_2 is isometric, but not surjective because $(W_2)^*(a)=0$. As $R_{S^*}\subset H_1$ we have $N_S=R_{S^*}^\perp\supset H_1^\perp\supset H_2$, so $S_2=0$. Now $D_2[W(I-S)-z]_2=$ $=[W(I-S)-z]_2D_2=I_2$, and $W_2-z=[W(I-S)-z]_2$ is invertible. However, Case 1 of the proof, applied to the operator W_2 in the separable Hilbert space H_2, yields $z\in\sigma(W_2)$. Contradiction.

As in Case 1, we find $\{z\in\mathbf{C}: |z|<1\}\subset\sigma(T)$. The theorem follows.

Theorem 9. *Let $T\in\mathscr{L}(H)$ and suppose that for all unitary $U\in\mathscr{L}(H)$ one has $\sigma(UT)=\{z\in\mathbf{C}: |z|\leqq\|T\|\}$. Then T is antinormal.*

Proof. For every unitary U one has $1+\|T\|\geqq\|T-U\|=\|U^*T-I\|\geqq 1+\|T\|$ because $-\|T\|\in\sigma(U^*T)$. Thus, $d(T,\mathscr{U}n)=1+\|T\|$ and T is antinormal according to Corollary 4.

Theorem 10. *Let $T\in\mathscr{L}(H)$,* ind $T<0$. *Let* $\omega:=\max(\aleph_0, \dim N_{T^*})$. *Then the following conditions are equivalent.*

(i) *T is antinormal.*

(ii) $d(T,\mathscr{U}n+\mathscr{C}_\omega)=1+\|T\|$.

(iii) $d(T,\mathscr{U}n+\mathscr{C})=1+\|T\|$.

Proof. Assume that T is antinormal. Then by Theorem 7 there exist a non-surjective isometry W with $\dim N_{W*} \leqq \dim N_{T*}$ and a positive Hermitian contraction $K \in \mathscr{C}_\omega$ such that $T = \alpha W(I-K)$ with $\alpha = \|T\| > 0$ and $\dim N_{W*} = \dim N_{T*}$ if $\dim N_{T*} \geqq \aleph_0$. Because $\alpha WK \in \mathscr{C}_\omega$ we have $d(T, \mathscr{U}n + \mathscr{C}_\omega) = d(\alpha W, \mathscr{U}n + \mathscr{C}_\omega) \leqq \leqq 1+\alpha$. Let $U \in \mathscr{U}n$, $P \in C_\omega$ and suppose $\|\alpha W - (U+P)\| < 1+\alpha$. If we can arrive at a contradiction, then Part (i)→(ii) of this theorem is proved. Distinguish two cases:

Case 1. Suppose N_{T*} is separable. We have $\|\alpha W - U\|_e \leqq \|\alpha W - U - P\| < < 1+\alpha$. Let $A := \alpha(1+\alpha)^{-1}(W+U)$. Then $\|U-A\|_e = (1+\alpha)^{-1}\|U-\alpha W\|_e < 1 = = m_e(U)$ and $\dim N_{U^*} = 0 < \aleph_0$, so $\dim N_{A*} < \aleph_0$ and $\operatorname{ind} A = \operatorname{ind} U = 0$ by Lemma 3 of [3]. Also, we have $\|\alpha W - A\|_e = \alpha(1+\alpha)^{-1}\|\alpha W - U\|_e < m_e(\alpha W)$, so that, by the same lemma, $\operatorname{ind} W = \operatorname{ind} A = 0$. Contradiction, as W is not surjective.

Case 2. Suppose N_{T*} is not separable. Because $P \in \operatorname{clo} \mathscr{C}_\omega$, there exists an S in $\mathscr{C}_\omega$ such that $\|W - (U+S)\| < 1+\alpha$. In the same way as in the proof of Case 2 of Theorem 8 one obtains a separable closed linear subspace H_1 of H that is invariant under αW, $(\alpha W)^*$, U, U^*, S and S^* whereas (using the subscript "1" to indicate restriction to H_1) W_1 is a non-surjective isometry and $S_1 = 0$. Using Case 1 we get $1+\alpha > \|\alpha W - (U+S)\| \geqq \|[\alpha W - (U+S)]_1\| = \|\alpha W_1 - U_1\| \geqq 1+\alpha$. Contradiction.

This proves the implication (i)→(ii). The remaining part of the theorem follows from Corollary 3 and the inequalities $d(T, \mathscr{U}n + \mathscr{C}_\omega) \leqq d(T, \mathscr{U}n + \mathscr{C}) \leqq d(T, \mathscr{U}n) \leqq \leqq 1 + \|T\|$.

Corollary 11. *Let $T \in \mathscr{L}(H)$, $\operatorname{ind} T < 0$. Let $\omega := \max(\aleph_0, \dim N_{T*})$. Then the following conditions are equivalent.*

(i) *T is antinormal.*

(ii) *T is essentially antinormal.*

(iii) $d(T, \mathscr{N} + \mathscr{C}_\omega) = \|T\|$.

Proof. Suppose T is antinormal. Let $N \in \mathscr{L}(H)$ be normal and $P \in \mathscr{C}_\omega$. There exist $\lambda > 0$ and unitary $U, V \in \mathscr{L}(H)$ with $N = \lambda(U+V)$. Then by Theorem 10,

$$\|T-(N+P)\| = \lambda\|\lambda^{-1}T - U - \lambda^{-1}P - V\| \geqq \lambda(\|\lambda^{-1}T - U - \lambda^{-1}P\| - 1) \geqq$$
$$\geqq \lambda(d(\lambda^{-1}T, \mathscr{U}n + \mathscr{C}_\omega) - 1) = \|T\|.$$

Thus, $d(T, \mathscr{N} + \mathscr{C}_\omega) = \|T\|$.

The implications (iii)→(ii) and (ii)→(i) are trivial.

Halmos ([6], Problem 113) has proved that the distance of the unilateral shift S to the set of normal operators in the Hilbert space l^2 is equal to 1. In his proof Halmos shows first that $d(S, \mathscr{I}nv) = \|S\|$. This equality does not only hold for the (antinormal) unilateral shift:

Theorem 12. *Let* $T \in \mathscr{L}(H)$, $\operatorname{ind} T < 0$ *and* $\omega := \max(\aleph_0, \dim N_{T^*})$. *Then the following conditions are equivalent.*

(i) T *is antinormal.*

(ii) $d(T, \mathscr{I}nv + \mathscr{C}_\omega) = \|T\|$.

(iii) $d(T, \mathscr{I}nv + \mathscr{C}) = \|T\|$.

(iv) $d(T, \mathscr{I}nv) = \|T\|$.

Proof. (i)→(ii). Without loss of generality, $\|T\| = 1$. By Theorem 7 there exist a non-surjective isometry $W \in \mathscr{L}(H)$ and a $K \in \mathscr{C}_\omega$ such that $T = W(I-K)$ and $\dim N_{W^*} \leqq \dim N_{T^*}$, while $\dim N_{W^*} = \dim N_{T^*}$ if N_{T^*} is infinite dimensional. Then $d(T, \mathscr{I}nv + \mathscr{C}_\omega) = d(W, \mathscr{I}nv + \mathscr{C}_\omega)$. Now suppose (ii) is false. Then there exist $G \in \mathscr{I}nv$ and $P \in \mathscr{C}_\omega$ with $\|W - (G+P)\| < 1$. Again we have to distinguish two cases.

Case 1. Suppose N_{T^*} is separable. We have $\|I - W^*G - W^*P\| \leqq \|W^*\| \cdot \|W - G - P\| < 1$, so $W^*G - W^*P$ is invertible, hence Fredholm, and $\operatorname{ind}(W^*G - W^*P) = 0$. Because W^*P is compact, W^*G is also Fredholm with index 0. (Theorem 5.20 of [2].) As G is invertible it follows that $\operatorname{ind} W^* = 0$, contradicting the fact that W is a non-surjective isometry.

Case 2. Suppose N_{T^*} is not separable. There exists an $S \in \mathscr{E}_\omega$ such that $\|W - (G+S)\| < 1$. Again, as in the proof of Case 2 of Theorem 8 there exists a separable closed linear subspace H_1 of H, invariant under $W, W^*, S, S^*, G, G^*, G^{-1}$ and $(G^{-1})^*$, such that the restriction of W to H_1 is a non-surjective isometry and S vanishes on H_1. An application of Case 1 to the restriction of $W - (G+S)$ to H_1 produces a contradiction.

This proves (i)→(ii). The implications (ii)→(iii) and (iii)→(iv) are trivial.

(iv)→(i). Suppose $d(T, \mathscr{I}nv) = \|T\|$. With the aid of the Spectral Theorem one easily proves that $\mathscr{I}nv \cap \mathscr{N}$ is dense in $\mathscr{N}$. Then $\|T\| = d(T, \mathscr{I}nv) \leqq d(T, \mathscr{I}nv \cap \mathscr{N}) = d(T, \mathscr{N}) \leqq \|T\|$, so $d(T, \mathscr{N}) = \|T\|$ and T is antinormal.

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